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Initial boundary value problem for the nonlinear Schrödinger equation

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Abstract. The integrable initial boundary value problem on a semi-line for the nonlinear Schrödinger equation is considered. It is shown that by means of the Bäcklund transformation this problem can be reduced to the well known Cauchy problem for the same equation on the line.

1. Introduction

Nowadays there is considerable interest in investigating boundary-value problems for integrable nonlinear partial differential equations [1–6]. In the traditional scheme of the inverse scattering transform (IST) [7, 8] for one-dimensional evolutionary systems the Cauchy problem on the line $x \in (-\infty, +\infty)$ is considered in the class of rapidly-decreasing functions at $x \to \pm\infty$. The possibility of efficiently investigating the Cauchy problem on the line is one of the main analytical achievements of IST. The periodical problem for integrable equations which led to the creation of the far advanced theory of finite-gap integration was another traditional object of investigation [7].

The investigation of the problem with local boundary conditions was much less simple. The first example (the nonlinear Schrödinger equation (NSE) on a semi-line with Dirichlet or Neumann boundary conditions) was studied in [9]. However, as a rule, it was impossible to apply the IST apparatus to solve such problems and the investigation was confined to numerical methods. A significant advance was made in [1] which suggested a method of obtaining boundary conditions consistent with the complete integrability of the model, and gave non-trivial examples of integrable boundary problems. For the investigation of the boundary problem on a semi-line for the NSE Sklyanin [1] suggested the use of a symmetrical reduction of the auxiliary linear problem which might naturally be called the NSE with point spin impurity. Such an approach was used in [2] to investigate a mixed boundary problem on a semi-line, and in [3] to construct algebro-geometric solutions of boundary problems on a semi-line and on an interval (see also [12]).

Habibullin [10] suggested another approach to the derivation of integrable boundary conditions based on the symmetrical reduction of the Bäcklund transformation (BT) associated with the original nonlinear equation. One can show both approaches to be equivalent. In particular, the impurity L-operator [2, 3] plays the role of the initial condition for a 'dressing-up' matrix in the BT (see section 2).

In this paper we will demonstrate the effectiveness of the second approach using the NSE in the attractive case

$$\mathrm{i}u_t + u_{xx} + 2|u|^2 u = 0 \tag{1.1}$$

and a mixed boundary condition at the point $x = 0$

$$(u_x - 2\alpha u)|_{x=0} = 0 \qquad (\alpha \in \mathbb{R}). \tag{1.2}$$

Our results may be easily transferred to other models allowing an integrable boundary condition to be obtained.

It should be noted that the third approach to boundary problems generally equivalent to the former ones was independently suggested in [4] the result of which has much in common with those of [2].

By the way, the integrable boundary problem for the NSE in the quantum case was investigated earlier than the classical one [11].

2. Derivation of boundary conditions from BT

Let us describe the way of obtaining boundary conditions by means of the 'dressing-up' procedure (or BT) using the example of a NSE-type complexified system

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} - 2u^2 v = 0 \\ &\mathrm{i}v_t - v_{xx} - 2v^2 u = 0. \end{aligned} \tag{2.1}$$

The linear spectral problem associated with (2.1) has the form [7],

$$\begin{aligned} &\Psi(x, \lambda) = U(x, \lambda)\Psi(x, \lambda) \\ &U(x, \lambda) = -\mathrm{i}\lambda\sigma_3 + \begin{pmatrix} 0 & v(x) \\ u(x) & 0 \end{pmatrix} \\ &\sigma_3 = \mathrm{diag}(1, -1). \end{aligned} \tag{2.2}$$

In addition to the initial 2×2 matrix Ψ-function, consider the dressed-up function $\tilde{\Psi}(x, \lambda)$:

$$\begin{aligned} &\tilde{\Psi}(x, \lambda) = L(x, \lambda)\Psi(x, \lambda) \qquad && L(x, \lambda) = (\lambda D(x) + A(x)) \\ &D \neq 0 && A(x) = a_{ij}(x) \qquad i, j = 1, 2 \end{aligned} \tag{2.3}$$

satisfying equation (2.2) with the substitution of the potentials $u(x)$ and $v(x)$ for the dressed-up potentials $\tilde{u}(x)$ and $\tilde{v}(x)$. The differential equation on matrix $L(x, \lambda)$ has the form:

$$L_x(x, \lambda) = \tilde{U}(x, \lambda)L(x, \lambda) - L(x, \lambda)U(x, \lambda). \tag{2.4}$$

Hence D is a constant diagonal matrix $D = \mathrm{diag}(d_1, d_2)$ and, besides, the algebraic relations:

$$\begin{aligned} &2\mathrm{i}a_{12} = d_2\tilde{u} - d_1 u \\ &2\mathrm{i}a_{21} = d_2 v - d_1 \tilde{v} \end{aligned} \tag{2.5}$$

and the differential equations:

$$\begin{aligned}\partial_x a_{11} &= \tilde{u} a_{21} - v a_{12}\\ \partial_x a_{22} &= \tilde{v} a_{12} - u a_{21}\\ \partial_x a_{12} &= \tilde{u} a_{22} - u a_{11}\\ \partial_x a_{21} &= \tilde{v} a_{11} - v a_{22}\end{aligned} \tag{2.6}$$

are valid. Note that det $L(x, \lambda) =$ constant, that is

$$d_1 a_{22} + d_2 a_{11} = \text{constant} \qquad a_{11}a_{22} - a_{12}a_{21} = \text{constant}. \tag{2.7}$$

From the system (2.5), (2.6) it follows that

$$\begin{aligned} d_2 \tilde{u}_x - d_1 u_x &= 2\mathrm{i}(\tilde{u} a_{22} - u a_{11})\\ d_1 \tilde{v}_x - d_2 v_x &= 2\mathrm{i}(v a_{22} - \tilde{v} a_{11}).\end{aligned} \tag{2.8}$$

It is easy to see that the system (2.5), (2.6) unambiguously determines (at least locally) matrix $A(x)$, and therefore $\tilde{u}(x)$, $\tilde{v}(x)$ at fixed $u(x)$, $v(x)$, D, $A(0)$. The pair $u(x)$, $v(x)$ may be called the BT from the functions $u(x)$ and $v(x)$.

The main observation is the following one. Suppose in relation (2.3) $D = I$ and we demand that the symmetry condition,

$$\begin{aligned} u(x) &= \tilde{u}(-x)\\ v(x) &= \tilde{v}(-x)\end{aligned} \tag{2.9}$$

should be satisfied. Then (see (2.8)) the boundary conditions

$$\begin{aligned}(u_x + \mathrm{i}(a_{22} - a_{11})u)|_{x=0} &= 0\\ (v_x + \mathrm{i}(a_{22} - a_{11})v)|_{x=0} &= 0\end{aligned}$$

are valid as in [1].

One can show that the system (2.5), (2.6) in the general case (see appendix A) is consistent with the symmetry relation (2.9) if and only if $d_1^2 = d_2^2 \neq 0$ and $d_1 a_{22} + d_2 a_{11} = 0$, special cases being of no interest. We shall confine ourselves to the case $d_1 = d_2 = 1$, $a_{11}(0) = -a_{22}(0) = -\mathrm{i}\alpha$, that is

$$L(0, \lambda) = \lambda - \mathrm{i}\alpha\sigma_3 \tag{2.10}$$

$$(u_x - 2\alpha u)|_{x=0} = 0 \qquad (v_x - 2\alpha v)|_{x=0} = 0. \tag{2.11}$$

The possibility of $d_1 = -d_2$ may be considered in a similar way (see appendix A).

This consideration allows the additional reduction of reality:

$$u(x) = -\bar{v}(x). \tag{2.12}$$

Under this condition the system (2.1) takes the form of NSE (1.1) and in formulae (2.3) we may take

$$\sigma_2 \overline{L(x, \bar{\lambda})} \sigma_2 = L(x, \lambda) \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix} \tag{2.13}$$

the spectral problem takes the form

$$\begin{aligned}\Psi_x(x, \lambda) &= U(x, \lambda)\Psi(x, \lambda)\\ U(x, \lambda) &= -\mathrm{i}\lambda\sigma_3 + \begin{pmatrix} 0 & u(x)\\ -\bar{u}(x) & 0\end{pmatrix}\end{aligned} \tag{2.14}$$

and in relations (2.10) and (2.11) α becomes real, so that (2.11) transforms into boundary condition (1.2). Further we shall confine ourselves to considering the real case (2.12). Another real reduction $u(x)=\bar{v}(x)$ will be studied in another paper.

Under reality conditions (2.12) the system (2.5), (2.6) will have a unique global solution at any permissible D and $A(0)$ (see appendix B). Furthermore we shall deal only with rapidly decreasing potentials $u(x): u(x)\to 0$, $|x|\to\pm\infty$. In this case the potential $\tilde{u}(x)$ will also be rapidly decreasing (see appendix C).

3. Boundary conditions and scattering data

Let us take $u(x)$ to be a smooth rapidly decreasing function on the line $x\in(-\infty,+\infty)$, satisfying the symmetry condition

$$u(x)=\tilde{u}(x). \tag{3.1}$$

We think that the dressing-up matrix $L(x,\lambda)$ satisfies relation (2.10). We shall show that condition (3.1) can be effectively formulated in terms of the scattering data for the potential $u(x)$.

Recall the definition of scattering data. Let $\Psi(x,\lambda)$ be arbitrary solution of x-equation (2.14) with the potential $u(x)$. The transition matrix $T(\lambda)$ is defined in the following way:

$$T(\lambda)=\lim_{x\to+\infty}(\exp(\mathrm{i}\lambda\sigma_3 x)\Psi(x,\lambda)\Psi^{-1}(-x,\lambda)\exp(\mathrm{i}\lambda\sigma_3 x)) \qquad (\operatorname{Im}\lambda=0) \tag{3.2}$$

and due to reduction (2.12) it has the form

$$T(\lambda)=\begin{pmatrix} a(\lambda) & -\overline{b(\lambda)} \\ b(\lambda) & \overline{a(\lambda)} \end{pmatrix} \qquad \det T(\lambda)=1.$$

An equivalent definition of $T(\lambda)$ as the connection matrix between Jost solutions $\Psi_\pm(x,\lambda)$:

$$\Psi_+(x,\lambda)=\Psi_-(x,\lambda)T(\lambda) \qquad \operatorname{Im}\lambda=0$$

follows from (3.2) with $\Psi(x,\lambda)=\Psi_-(x,\lambda)$. (Remember that $\Psi_\pm(x,\lambda)$ are the solutions of equation (2.14) with the following asymptotics $\Psi_\pm(x,\lambda)\to\exp(-\mathrm{i}\lambda\sigma_3 x)$ if $x\to\pm\infty$.)

It is a matter of common knowledge [8] that in the case of rapidly decreasing functions $u(x)$ the coefficient $a(\lambda)$ has an analytical continuation into the upper half-plane $\mathbb{C}$ and probably has zeros at points λ_j, $\operatorname{Im}\lambda_j>0$ there. The Jost function columns at these points are proportional to each other:

$$\gamma(\lambda_j)\Psi_+^2(\lambda_j)=\Psi_-^2(\lambda_j) \qquad \gamma(\lambda_j)\neq 0,\infty. \tag{3.3}$$

Let us consider all the zeros $a(\lambda)$ to be simple and non-real and their number to be finite. We shall denote the set $S(\lambda)=\{b(\lambda),\lambda_j,\gamma(\lambda_j), j=1,\ldots,n\}$ of scattering data for the potential $u(x)$. The coefficient $a(\lambda)$ is obtained at $\operatorname{Im}\lambda\geqslant 0$ by the formula

$$a(\lambda)=\prod_{k=1}^{n}\frac{\lambda-\lambda_k}{\lambda-\bar{\lambda}_k}\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\log|1-|b(\eta)|^2|}{\eta-\lambda}\,\mathrm{d}\eta\right). \tag{3.4}$$

We know [8] that between the scattering data $S(\lambda)$ and the potential $u(x)$ there is a one-to-one correspondence.

Proposition 1. The symmetry (3.1) of the potential $u(x)$ is equivalent to the following restriction on scattering data $S(\lambda)$:

$$\begin{cases} a(-\lambda) = \overline{a(\bar{\lambda})} \\ b(-\lambda) = b(\lambda)\dfrac{\lambda - \mathrm{i}\beta}{\lambda + \mathrm{i}\beta} \qquad \beta = \alpha(-1)^n \end{cases} \tag{3.5}$$

$$\gamma(-\lambda_k)\overline{\gamma(-\bar{\lambda}_k)} = \frac{\lambda_k + \mathrm{i}\beta}{\lambda_k - \mathrm{i}\beta}. \tag{3.6}$$

Proof. Let us first prove the statement (3.5). From the symmetry (3.1) it follows that $\sigma_3 \tilde{U}(-x, -\lambda)\sigma_3 = -U(x, \lambda)$, hence

$$\sigma_3 \tilde{\Psi}(-x, -\lambda) = \Psi(x, \lambda) M(\lambda) \tag{3.7}$$

with some matrix $M(\lambda)$ independent on x. From the definition (3.2) of the scattering matrix and formula (2.3) it follows that

$$\tilde{T}(\lambda) = \sigma_3 T^{-1}(-\lambda)\sigma_3 = (\lambda - \mathrm{i}\beta_+\sigma_3) T(\lambda)(\lambda - \mathrm{i}\beta_-\sigma_3)^{-1} \tag{3.8}$$

where $(\lambda - \mathrm{i}\beta_\pm\sigma_3) = L(\pm\infty, \lambda)$ are the limiting values of dressing-up L-matrix, $\beta_\pm^2 = \alpha^2$, (see appendix C).

Lemma 1. The relation

$$\beta_+ = \beta_- \equiv \beta \tag{3.9}$$

is valid.

Proof. Using the condition (3.7) twice and taking (2.3) into account, we shall get the equality

$$\sigma_3 L(-x, -\lambda)\sigma_3 L(x, \lambda) = M(\lambda) M(-\lambda). \tag{3.10}$$

Taking $x = 0$ and taking into account (2.10) we have $M(\lambda)M(-\lambda) = -(\lambda^2 + \alpha^2)I$. The variable x tending to $\pm\infty$ in (3.10) we get relation (3.9). Lemma 1 is proved.

Lemma 2. Let n be the number of points of the discrete spectrum λ_j, $\operatorname{Im}\lambda_j > 0$ for the potential $u(x)$. Then $\beta = (-1)^n \alpha$.

Proof. From the relationships (2.10) and (3.7) it follows that $M(0) = -\mathrm{i}\alpha I$. Now we shall use the representation following from (3.7) $\Psi(x, \lambda) = L^{-1}(x, \lambda)\sigma_3 \Psi(-x, -\lambda) M(-\lambda)$, and calculate matrix $T(\lambda)$ in the point $\lambda = 0$: $T(0) = (\alpha/\beta)I$. On the other hand, from (3.8) and (3.9) it follows that $a(0) = \pm 1$, $b(0) = 0$, $T(0) = a(0)I$. From the representation (3.4) for $a(\lambda)$ it follows that $a(0) = (-1)^n$, that is the statement of lemma 2 is proved. The formulae (3.5) are readily derived from (3.8) and lemmas 1 and 2.

Now we shall prove formula (3.6). From the definition (3.3), the Jost function property $\Psi_+(x, \lambda) = \sigma_3 \tilde{\Psi}_-(-x, -\lambda)\sigma_3$, and normalization condition (2.10) we get at $x = 0$

$$\Psi_+^1(0, -\lambda_k) = -\gamma(\lambda_k) \frac{\lambda_k - i\beta}{\lambda_k + i\beta} \Psi_-^2(0, -\lambda_k). \tag{3.11}$$

On the other hand due to reality condition (2.12)

$$\overline{\Psi_\pm(x, \bar{\lambda})} = \sigma_2 \Psi_\pm(x, \lambda)\sigma_2$$

and therefore

$$\Psi_-^2(0, -\lambda_k) = -\overline{\gamma(-\bar{\lambda}_k)}\Psi_+^1(0, -\lambda_k). \tag{3.12}$$

Comparing equations (3.11) and (3.12) we come to formula (3.11). One can see that due to one-to-one correspondence $S(\lambda) \leftrightarrow u(x)$ our consideration can be reversed, that is the symmetry property (3.1) of the potential $u(x)$ can be derived from the formulae (3.5) and (3.6). Proposition 1 is proved. □

From formula (3.6) it follows that purely imaginary points of the discrete spectrum λ_j, $\mathrm{Re}\,\lambda_j = 0$ may lie only beyond the interval $[0, i|\alpha|]$, i.e. $|\lambda_j| > |\alpha|$. Note that for the finite potential $u(x)$ formula (3.6) can be obtained by analytical continuation from formulae (3.5).

4. Boundary problem for NSE on a semi-line

Let $u_0(x)$ be a smooth function at $x \geqslant 0$, rapidly decreasing at $x \to +\infty$ and satisfying boundary condition (1.2) at $x = 0$. Then we make the BT $\tilde{u}_0(x)$ of the function $u_0(x)$ with the dressing-up matrix $L(x, \lambda)$, normalized by the condition (2.10). Note, that due to (2.5), (2.6) and (2.10)

$$\tilde{u}_0(0) = u_0(0) \qquad \partial_x \tilde{u}_0(0) = -\partial_x u_0(0) \qquad \partial_x^2 \tilde{u}_0(0) = \partial_x^2 u_0(0).$$

Now let us introduce the function $u(x)$ defined on the line

$$u(x) = u_0(x)H(x) + \tilde{u}_0(-x)H(-x) \qquad x \in \mathbb{R} \tag{4.1}$$

where $H(x)$ is the Heaviside function: $H(x) = (1 + \mathrm{sign}(x))/2$. We can see that $u(x)$ is a C^2-smooth function rapidly decreasing at $|x| \to \infty$ and satisfying symmetry (3.1) by construction. The higher smoothness $u(x)$ is equivalent to validity of additional boundary conditions on the function $u_0(x)$ (see appendix D).

Now we have all that is necessary to solve the following initial boundary problem for the NSE on a semi-line: the construction of the function $u(x, t)$, satisfying equation (1.1) at $x \geqslant 0$, boundary condition (1.2) and initial condition $u(x, 0) = u_0(x)$. To solve this problem let us consider the function $u(x)$ from (4.1) as the initial datum for NSE (1.1) on the line; let $u(x, t)$ be a solution of such a Cauchy problem. The scattering data for the potential $u(x, t)$ depend on time in a well known way [8]:

$$\begin{aligned} a(\lambda, t) &= a(\lambda, 0) \\ b(\lambda, t) &= b(\lambda, 0)\exp(4i\lambda^2 t) \\ \gamma(\lambda_j, t) &= \gamma(\lambda_j, 0)\exp(4i\lambda_j^2 t) \end{aligned} \tag{4.2}$$

and at $t=0$ satisfy the symmetry conditions (3.5) and (3.6). However, these symmetries are clearly compatible with evolution (4.2), hence $u(x, t)$ satisfies relation (3.1) and therefore boundary condition (1.2) at all t. Consequently, the restriction of $u(x, t)$ on the positive semi-line gives the solution of the initial boundary value in question at all t.

Besides, returning to complexified system (2.1), let us consider its solution $u(x, t)$, $v(x, t)$. Requiring that the BT $\tilde{u}(x, t)$, $\tilde{v}(x, t)$ should also be a solution of the system (2.1) we shall obtain the equation

$$L_t(x, t, \lambda) = \tilde{V}(x, t, \lambda) L(x, t, \lambda) - L(x, t, \lambda) V(x, t, \lambda)$$

$$V(x, t, \lambda) = 2\lambda U(x, t, \lambda) + \begin{pmatrix} u(x, t)v(x, t) & -u_x(x, t) \\ v_x(x, t) & -u(x, t)v(x, t) \end{pmatrix}$$

which is consistent with equation (2.4). Thereby the BT is transferred to solutions of system (2.1) and the symmetry relations (2.9) can easily be seen to agree with the dynamics over t. Therefore the suggested method of solving the initial boundary value problem in the class of rapidly decreasing functions can be extended to other cases (e.g. the class of functions having constant non-zero asymptotics, i.e. those satisfying the so-called boundary conditions of 'finite density').

Appendix A

Let us determine the conditions at which the system (2.5), (2.6) is compatible with the symmetry relation (2.9). We shall denote $\tilde{A}(x) = A(-x)$, and write down the equations, which follows from (2.6) and (2.8) after substitution $(x \to -x)$ and account for the relation (2.9)

$$\partial_x \tilde{a}_{11} = \tilde{v} a_{12} - u \tilde{a}_{21}$$

$$\partial_x \tilde{a}_{22} = \tilde{u} \tilde{a}_{21} - v \tilde{a}_{12}$$

$$\partial_x \tilde{a}_{12} = \tilde{u} \tilde{a}_{11} - u \tilde{a}_{22}$$

$$\partial_x \tilde{a}_{21} = \tilde{v} \tilde{a}_{22} - v \tilde{a}_{11}$$

$$d_2 u_x - d_1 \tilde{u}_x = 2\mathrm{i}(\tilde{u} \tilde{a}_{11} - u \tilde{a}_{22})$$

$$d_2 \tilde{v}_x - d_1 v_x = 2\mathrm{i}(\tilde{v} \tilde{a}_{22} - v \tilde{a}_{11}).$$

These equations together with the equations (2.6) and (2.8) form the system $M \cdot \xi_x = F(\xi)$ for vector

$$\xi(x) = (u(x), v(x), \tilde{u}(x), \tilde{v}(x), \tilde{A}(x), \tilde{A}(x)) \in \mathbb{C}^{12}$$

where $F(\xi)$ is a smooth function. If $d_1^2 \neq d_2^2$ the matrix M is reversible and, at least near the point $x=0$, $\xi(x)$ is unambiguously determined by the initial value $\xi(0)$. Finally we shall get the six-parametric class of potentials $u(x)$, $v(x)$ which is not obviously sufficient for the solution of the initial boundary value problem.

Now let $d_1 = d_2$. Due to (2.5), $a_{12}(x)$ and $a_{21}(x)$ are odd functions, therefore due to (2.6) $u(x)a_{11}(x) - \tilde{u}(x)a_{22}(x)$, $\tilde{v}(x)a_{11}(x) - v(x)a_{22}(x)$ and $a_{11}(x) - a_{22}(x)$ are even functions, hence either $a_{11}(x) + a_{22}(-x) = 0$ or $u(x) = \tilde{u}(-x)$ and $v(x) = \tilde{v}(-x)$. In the general case it is enough to consider only the first possibility because it contains the case of odd potentials $u(x)$, $v(x)$, as well. Similarly, in the case $d_1 = -d_2$ we get the relation $a_{11}(x)-a_{22}(-x)=0$. It also is clear that at $d_1^2=d_2^2$ and $d_1a_{11}(0)+d_2a_{22}(0)=0$ the system (2.5), (2.6) is indeed consistent with the symmetry relation (2.9).

If $d_1=-d_2$ equation (2.8) becomes an identity. Differentiating (2.8) over x substituting $x=0$ we get boundary conditions

$$(u_{xx}-\mathrm{i}(a_{11}+a_{22})u_x)|_{x=0}=0$$

$$(v_{xx}-\mathrm{i}(a_{11}+a_{22})v_x)|_{x=0}=0.$$

Appendix B

We shall show that, in the real case (2.12), equations (2.5) and (2.6) are globally solvable for any admissible (see (2.13)) initial data $L(0,\lambda)$. Because matrix $D\neq 0$ and $\sigma_2\bar{D}\sigma_2=D$ then D is a reversible matrix $D=\mathrm{diag}(d,\bar{d})$. The dressing-up matrix $L(\lambda)$ can be represented in the form

$$L(\lambda)=D\cdot(\lambda+D^{-1}A)=D\cdot(\lambda+\lambda_0+A_0)$$

where $\lambda_0=$ constant, $\mathrm{Tr}\,A_0=0$, $A_0=-A_0^\dagger$. In order not to complicate the presentation by technical details we shall closely consider only the case $D=I$, $A(0)=-\mathrm{i}\alpha\sigma_3$, which is just necessary for our purpose. The general case is similarly analysed.

Let $\Psi(x,\lambda)$ be the solution of equation (2.14) corresponding to the potential $u(x)$ and satisfying the reduction

$$\sigma_2\Psi(x,\lambda)\sigma_2=\Psi(x,\lambda) \tag{B1}$$

for whose validity it is enough to choose $\Psi(0,\lambda)=I$. Matrix $\tilde{\Psi}(x,\lambda)$ (see (2.3)) due to (2.7) and (2.10) is degenerate at points $\lambda=\pm\mathrm{i}\alpha$ at all x. According to (1.2) the subspaces $\ker\tilde{\Psi}(0,\pm\mathrm{i}\alpha)$ are produced by vectors $\binom{1}{0}$ and $\binom{0}{1}$ respectively. Let us require that these subspaces should be the same at all x

$$\begin{cases}\tilde{\Psi}(x,\mathrm{i}\alpha)\begin{pmatrix}1\\0\end{pmatrix}=0\\[2ex]\tilde{\Psi}(x,-\mathrm{i}\alpha)\begin{pmatrix}0\\1\end{pmatrix}=0.\end{cases} \tag{B2}$$

Equation (B2) is the system for determination of matrix $A(x)$

$$\begin{cases}A\Psi^1(x,\mathrm{i}\alpha)=\mathrm{i}\alpha\Psi^1(x,\mathrm{i}\alpha)\\A\Psi^2(x,-\mathrm{i}\alpha)=-\mathrm{i}\alpha\Psi^2(x,-\mathrm{i}\alpha).\end{cases} \tag{B3}$$

(Ψ^1 and Ψ^2 are the columns of matrix $\Psi=(\Psi^1,\Psi^2)$.) Note that due to the reduction (B1) eigenvectors of matrix $A(x)$ are orthogonal in $\mathbb{C}^2$ at all x. So the system (B3) is always uniquely solvable and determines the function $\tilde{u}(x)$ (see (2.5)).

Due to the fact that the degeneration points of $\tilde{\Psi}$-function $\lambda=\pm\mathrm{i}\alpha$ and corresponding subspaces $\ker\tilde{\Psi}(x,\pm\mathrm{i}\alpha)$ do not depend on x, the function $\tilde{\Psi}(x,\lambda)$ satisfies (see [8]) equation (2.14) with the dressed-up potential $\tilde{u}(x)$, equations (2.5) and (2.6) being valid for matrix $A(x)$. Thus we have the global solution of the system (2.5), (2.6) with arbitrary admissible initial data $A(0)$. The uniqueness of such solution follows from the general theory of ordinary differential equations.

Appendix C

We shall show that if the potential $u(x)$ is rapidly decreasing $u(x)\to 0$, $|x|\to\infty$, then its BT $\tilde{u}(x)$ is also rapidly decreasing and the dressing-up matrix $L(x, \lambda)$ has diagonal asymptotics

$$L(x,\lambda)\to\lambda D+A_\pm \qquad x\to\pm\infty.$$

Again we shall only confine ourselves to an interesting case (2.10). Let $\Psi(x, \lambda)$ be a matrix consisting of the Jost functions columns

$$\Psi(x,\lambda)=\begin{cases}(\Psi_-^1,\Psi_+^2)(x,\lambda) & \operatorname{Im}\lambda\geqslant 0\\ (\Psi_+^1,\Psi_-^2)(x,\lambda) & \operatorname{Im}\lambda\leqslant 0.\end{cases}$$

The Ψ-function obtained is analytical on λ in the upper and lower half-planes, and at $x\to\pm\infty$, $|\operatorname{Im}\lambda|>0$ has asymptotics (see [8])

$$\Psi(x,\lambda)\to(C_\pm(\lambda)+\mathrm{O}(1))\exp(-\mathrm{i}\lambda\sigma_3 x) \tag{C1}$$

with some diagonal matrices $C_\pm(\lambda)$ not depending on x.

The dressed-up matrix $\tilde{\Psi}(x, \lambda)$ degenerates in points $\lambda=\pm\mathrm{i}\vartheta$, $\vartheta=|\alpha|$, subspaces $\ker\tilde{\Psi}(x, \pm\mathrm{i}\vartheta)$ being produced by either

$$\begin{pmatrix}1\\c\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}-\bar{c}\\1\end{pmatrix} \qquad c\neq\infty$$

or

$$\begin{pmatrix}0\\1\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}1\\0\end{pmatrix}$$

respectively. Let $x\to\pm\infty$. Then taking into account (C1) we come to asymptotic equalities

$$\left.\begin{aligned}&L(+\infty,\mathrm{i}\vartheta)\begin{pmatrix}1\\0\end{pmatrix}=0\\ &L(-\infty,-\mathrm{i}\vartheta)\begin{pmatrix}0\\1\end{pmatrix}=0\end{aligned}\right. \qquad \text{in the first case}$$

and

$$\left.\begin{aligned}&L(+\infty,\mathrm{i}\vartheta)\begin{pmatrix}0\\1\end{pmatrix}=0\\ &L(-\infty,-\mathrm{i}\vartheta)\begin{pmatrix}1\\0\end{pmatrix}=0\end{aligned}\right. \qquad \text{in the second case.}$$

Hence

$$L(+\infty,\lambda)=\lambda-\mathrm{i}\beta_+\sigma_3 \qquad \beta_+^2=\alpha^2. \tag{C2}$$

Similarly we find that at $x\to-\infty$

$$L(-\infty,\lambda)=\lambda-\mathrm{i}\beta_-\sigma_3 \qquad \beta_-^2=\alpha^2. \tag{C3}$$

From (C2), (C3) it immediately follows that the potential $\tilde{u}(x)$ is rapidly decreasing (see (2.5)).

Appendix D

As we have mentioned, the validity of boundary condition (1.2) for $u_0(x)$ provides C^2-smoothness of the function $u(x)$ determined by formula (4.1). The requirement of higher smoothness of $u(x)$ is equivalent to additional boundary conditions on $u_0(x)$ of a higher order than the derivatives. They are obtained by means of differentiating the equalities (2.7) with the subsequent substitution $x=0$. The derivatives from other functions can be eliminated using equalities (2.5), (2.6) and (2.9) and their implications. The same boundary conditions may be obtained by differentiation of boundary condition (1.2) over t and using NSE (1.1). These additional boundary conditions are also boundary conditions for higher NSEs; they were considered in such terms in [2]. For example, the boundary condition

$$(\partial_x^3 u_0 - 2\alpha(\partial_x^2 u_0 + 4|u_0|^2 u_0))|_{x=0} = 0$$

is equivalent to C^4-smoothness of $u(x)$.

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